The Geometric Satake Equivalence

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§1. Introduction

Where we are now

- k algebraically closed field, G reductive group over k.
- The *Satake category* Sat_G is the category of L^+G-equivariant $\overline{\mathbf{Q}}_\ell$-perverse sheaves on the affine Grassmannian $\mathrm{Gr} = \mathrm{L}G/\mathrm{L}^+G$.
- The Satake category is a semi-simple abelian category and admits a symmetric monoidal tensor product $\star$.
- There is a symmetric monoidal functor $\mathrm{H}^*\colon\ \mathsf{Sat}_G \to \mathsf{Vec}$ given by taking the sum of all hypercohomology groups.

Goals for this talk

- Identify Sat_G with $\mathrm{Rep}(G^\vee)$, where $G^\vee$ is the Langlands dual group. This is the *geometric Satake equivalence*.
- Explain how this categorifies the classical Satake isomorphism.

§2. The Langlands dual group

Root systems

Definition

A *root system* is a pair (V, Φ) where V is a finite dimensional real vector space equipped with a positive definite inner product $(,)$ and Φ is a finite subset of V such that:

- Φ spans V and does not contain 0.
- For $\alpha, \beta \in \Phi$, the number $2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer.
- For $\alpha, \beta \in \Phi$, the element $\beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha$ belongs to Φ.

We say the root system is *reduced* if no scalar multiple of a root (other than ± 1) is a root.

Root systems and Lie algebras

- Let $\mathfrak{g}$ be a complex semisimple Lie algebra.
- Let $\mathfrak{h}$ be a Cartan subalgebra.
- Have a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ where Φ is a finite subset of $\mathfrak{h}^*$ and $\mathfrak{g}_\alpha$ is the α-eigenspace of $\mathfrak{h}$ acting on $\mathfrak{g}$.
- The Killing form induces a non-degenerate symmetric bilinear form on $\mathfrak{h}$ (and thus $\mathfrak{h}^*$).
- Get root system by taking V to be the $\mathbf{R}$-subspace of $\mathfrak{h}^*$ spanned by Φ.
- The root system determines $\mathfrak{g}$ up to isomorphism, and every (reduced) root system comes from some $\mathfrak{g}$.

An example

Let $\mathfrak{g}$ be the complex semisimple Lie algebra $\mathfrak{sl}_2$. This has for a basis elements X, Y, and H, and brackets

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

Can take $\mathfrak{h}$ to be the span of H. Identifying $\mathfrak{h}^*$ with $\mathbf{C}$, we have $\Phi = \{2, -2\}$.

Root data

Definition

A *root datum* consists of data $(X^\bullet, \Phi, X_\bullet, \Phi^\vee)$ where:

- $X^\bullet$ and $X_\bullet$ are free abelian groups of finite rank equipped with a perfect pairing $(,)\colon X^\bullet \otimes X_\bullet \to \mathbf{Z}$.
- Φ is a finite subset of $X^\bullet$, $\Phi^\vee$ is a finite subset of $X_\bullet$, and there is a given bijection $\Phi \to \Phi^\vee$ denoted $\alpha \mapsto \alpha^\vee$.
- For $\alpha \in \Phi$ we have $(\alpha, \alpha^\vee) = 2$.
- For $\alpha \in \Phi$, the map $x \mapsto x - (x, \alpha^\vee)\alpha$ carries Φ into itself, and the dual map carries $\Phi^\vee$ into itself.

Elements of Φ are called *roots*, and elements of $\Phi^\vee$ *coroots*.

Given a root datum, get a root system by taking $V = X^\bullet \otimes \mathbf{R}$.

Root data and algebraic groups

- Let G be a connected reductive algebraic group over an algebraically closed field with maximal torus T.
- Define $X^\bullet = \mathrm{Hom}(T, \mathbf{G}_m)$ and $X_\bullet = \mathrm{Hom}(\mathbf{G}_m, T)$. We call $X^\bullet$ the *weight lattice* and $X_\bullet$ the *coweight lattice*.
- Define Φ to be the set of non-zero weights appearing in the T action on $\mathrm{Lie}(G)$.
- For each root α, there is a unique coweight $\alpha^\vee \colon \mathbf{G}_m \to T$ satisfying some conditions (one being $\alpha(\alpha^\vee(t)) = t^2$). Take $\Phi^\vee$ to be the set of these $\alpha^\vee$'s.
- This defines a root datum, which determines G up to isomorphism. Every (reduced) root datum comes in this way.

Example 1

- Take $G = \mathrm{SL}(2)$ and T to be the diagonal torus.
- Identify $X^\bullet$ and $X_\bullet$ with $\mathbf{Z}$.
- Have $\Phi = \{\pm 2\}$.
- This forces $\Phi^\vee = \{\pm 1\}$.
- Note that $\langle \Phi \rangle$ has index 2 in $X^\bullet$.

Example 2

- Now take $G' = \mathrm{PSL}(2)$ and T' to be the diagonal torus.
- The surjection $G \to G'$ identifies $X^\bullet(G')$ with $2\mathbf{Z} \subset X^\bullet(G)$.
- Since this surjection identifies Lie algebras (char. $\neq 2$), it identifies roots. Thus $\Phi(G') = \{\pm 2\} \subset 2\mathbf{Z}$.
- Similarly, $\Phi^\vee(G') = \{\pm 1\} \subset \frac{1}{2}\mathbf{Z}$.
- Note that $\langle \Phi(G') \rangle = X^\bullet(G')$. Thus the root datum for G' is not isomorphic to the one for G.

Some properties of the root datum

One can easily read off some properties of G from its root datum:

- G is a torus if and only if $\Phi = \emptyset$.
- G is semisimple $\Leftrightarrow$ $\langle \Phi \rangle$ is a finite index in $X^{\bullet}$.
- $\pi_1(G) = X_{\bullet}/\langle \Phi^{\vee} \rangle$.
- $\mathrm{Hom}(Z(G), \mathbf{G}_m) = X^{\bullet}/\langle \Phi \rangle$.

(For last two points, assume characteristic is 0 or large.)

The dual group

Let G be a connected reductive group over an algebraically closed field k. Then G is classified by a root datum

$$(X^\bullet, \Phi, X_\bullet, \Phi^\vee)$$

We define the *dual group* $G^\vee$ to be the unique (up to isomorphism) connected reductive group with root datum

$$(X_\bullet, \Phi^\vee, X^\bullet, \Phi).$$

One can define $G^\vee$ over any field; we'll be interested in taking $G^\vee$ over $\overline{\mathbf{Q}}_\ell$.

Examples of dual groups

G	$G^\vee$
$\mathrm{GL}(n)$	$\mathrm{GL}(n)$
$\mathrm{SL}(n)$	$\mathrm{PSL}(n)$
$\mathrm{SO}(2n+1)$	$\mathrm{Sp}(2n)$
$\mathrm{SO}(2n)$	$\mathrm{SO}(2n)$
$\mathrm{Spin}(2n)$	$\mathrm{SO}(2n)/\{\pm 1\}$

§3. Tannakian duality

Rigid tensor categories

Let k be a field and let $\mathcal{C}$ be a k-linear symmetric tensor category with unit object $\mathbf{1}$.

We say that an object X is *dualizable* if there exists an object $X^\vee$ and a map $X \otimes X^\vee \to \mathbf{1}$ such that for any other object T, the natural map

$$\mathrm{Hom}(T, X^\vee) \to \mathrm{Hom}(T \otimes X, \mathbf{1})$$

is an isomorphism.

Definition

We say that $\mathcal{C}$ is *rigid* if all objects admit a dual.

Examples

- Let G be an affine group scheme over k. Then the category $\mathrm{Rep}(G)$ of finite dimensional representations of G is a rigid tensor category. (Here "representation" means "comodule over $k[G]$.") This is the motivating example.
- The Satake category Sat_G (as we'll see).
- The category of pure Hodge structures.
- The category of pure motives over a field (conjecturally).

Fiber functors

Let $\mathcal{C}$ be a k-linear rigid tensor category.

Definition

A *fiber functor* on $\mathcal{C}$ is a faithful exact k-linear tensor functor $\omega\colon \mathcal{C} \to \mathsf{Vec}$.

Given a fiber functor ω, we define a functor $\underline{\mathrm{Aut}}(\omega)$ on k-algebras by associating to a k-algebra R the group of automorphisms of the tensor functor $\omega\colon \mathcal{C} \otimes_k R \to \mathsf{Mod}_R$.

Fiber functors: motivation

Suppose $\mathcal{C} = \mathrm{Rep}(G)$ for an affine group scheme G/k. We then have a natural fiber functor $\omega\colon \mathrm{Rep}(G) \to \mathrm{Vec}$, namely, the forgetful functor.

Given any $V \in \mathrm{Rep}(G)$ and any element $g \in G(R)$, we have a given map $g\colon V \otimes R \to V \otimes R$. This defines a map $i\colon G \to \underline{\mathrm{Aut}}(\omega)$ of group-valued functors.

Theorem (Tannaka reconstruction)

The map i is an isomorphism.

This theorems shows how to reconstruct G from $(\mathcal{C}, \omega)$.

The main theorem

Theorem

Let $\mathcal{C}$ be a k-linear rigid tensor category with $\mathrm{End}(\mathbf{1}) = k$, and let ω be a fiber functor on $\mathcal{C}$. Then:

(a) *The functor $\underline{\mathrm{Aut}}(\omega)$ is represented by an affine group scheme G; and*

(b) *The natural functor $\mathcal{C} \to \mathrm{Rep}(G)$ is an equivalence of tensor categories.*

Relating G and $\mathrm{Rep}(G)$

Given $(\mathcal{C}, \omega)$ as in the previous slide, we would like to translate information about $\mathcal{C}$ to information about the group G. By the theorem, it suffices to consider $\mathcal{C} = \mathrm{Rep}(G)$.

- G is finite over k if and only if there is some $X \in \mathrm{Rep}(G)$ such that every representation is a subquotient of $X^{\oplus n}$ for some n.
- G is of finite type over k if and only if there is some $X \in \mathrm{Rep}(G)$ such that every representation is a subquotient of a finite sum of representations of the form $X^{\otimes n}$.

Relating G and $\mathrm{Rep}(G)$ (cont)

In what follows, we assume k has characteristic 0 and G is of finite type over k.

- G is disconnected if and only if there exists a non-trivial representation X such that the subcategory of $\mathrm{Rep}(G)$ spanned by subquotients of $X^{\oplus n}$ is stable under $\otimes$. (Idea: take X to be regular representation of $\pi_0(G)$.)
- Assume G is connected. Then G is reductive if and only if $\mathrm{Rep}(G)$ is semisimple.

§4. Geometric Satake (part 1)

Setup

- Fix an algebraically closed field k and a reductive group G/k.
- Recall that the Satake category Sat_G is the category of L^+G-equivariant perverse $\overline{\mathbf{Q}}_\ell$-sheaves on $\mathrm{Gr} = \mathrm{L}G/\mathrm{L}^+G$.
- We let $\star$ be the convolution product on Sat_G.
- For a dominant coweight μ, let Gr_μ be the L^+G-orbit of t^μ, and $\mathrm{Gr}_{\leq\mu}$ its closure. Let IC_μ be the IC sheaf of Gr_μ.
- The category Sat_G is semisimple, and its simple objects are the IC_μ.

A lemma on intersection cohomology

Lemma

Let X be an irreducible projective variety of dimension n and let $H = \bigoplus_{i \geq 0} \mathrm{H}^i(X, \mathrm{IC}_X)$. Then $\dim(H) \geq 1 + n$.

Proof

- Let $f \colon X \to \mathbf{P}^n$ be a generically finite map.
- By the decomposition theorem, $f_*(\mathrm{IC}_X)$ is a direct sum of shifts simple perverse sheaves.
- One of the summands is $\mathrm{IC}_{\mathbf{P}^n}$ (on appropriate open subsets we get the constant $\overline{\mathbf{Q}}_\ell$-sheaf as a summand).
- We thus see that $\bigoplus_{i \geq 0} \mathrm{H}^i(\mathbf{P}^n, \mathrm{IC}_{\mathbf{P}^n})$ is a summand of H, and this has dimension $n + 1$. (Note that $\mathrm{IC}_{\mathbf{P}^n}$ is a shift of a constant sheaf since $\mathbf{P}^n$ is smooth.)

Zero dimensional orbits

Let μ be a dominant coweight. The following are equivalent:

(a) μ is central.

(b) t^μ normalizes $\mathrm{L}^+ G$ inside of $\mathrm{L}G$.

(c) t^μ is fixed by $\mathrm{L}^+ G$.

(d) Gr_μ is zero-dimensional.

(e) $\mathrm{H}^*(\mathrm{IC}_\mu)$ is one-dimensional.

If μ and λ are central dominant coweights then a simple computation shows that $\mathrm{IC}_\mu \star \mathrm{IC}_\lambda \cong \mathrm{IC}_{\mu+\lambda}$. (We'll also deduce this from a more general result below.)

Sat_G as a rigid tensor category

Proposition

Sat_G *is a rigid tensor category and* H^* *is a fiber functor.*

Proof

- We know that Sat_G is a symmetric tensor category and that H^* is a tensor functor.
- H^* is automatically exact since Sat_G is semisimple.
- By the lemma, $\mathrm{H}^*(\mathrm{IC}_\mu)$ is non-zero. Since Sat_G is semisimple, this implies that H^* is faithful.
- To show Sat_G is rigid, it suffices (by Deligne–Milne) to show that if $\mathrm{H}^*(\mathrm{IC}_\mu)$ is one-dimensional then IC_μ is invertible. If $\mathrm{H}^*(\mathrm{IC}_\mu)$ is one-dimensional then μ is central, and $\mathrm{IC}_\mu \star \mathrm{IC}_{-\mu} \cong \mathrm{IC}_0 = \mathbf{1}$ (by previous slide).

Sat_G as a rigid tensor category (cont)

Applying the main theorem of Tannakian duality, we find:

Corollary

There is an affine group scheme $\Gamma/\overline{\mathbf{Q}}_\ell$ *such that* Sat_G *is equivalent to* $\mathrm{Rep}(\Gamma)$.

This equivalence is one of $\overline{\mathbf{Q}}_\ell$-linear symmetric tensor categories, and respects the fiber functors.

To prove the geometric Satake equivalence, we now simply need to identify Γ as the Langlands dual of G.

A lemma on convolutions

Lemma

Let λ and μ be dominant coweights. Then $\mathrm{IC}_\lambda \star \mathrm{IC}_\mu$ contains $\mathrm{IC}_{\lambda+\mu}$ as a summand.

Proof

- Consider the diagram

$$\mathrm{Gr} \times \mathrm{Gr} \xleftarrow{p} \mathrm{L}G \times \mathrm{Gr} \xrightarrow{q} \mathrm{L}G \times^{\mathrm{L}^+G} \mathrm{Gr} \xrightarrow{m} \mathrm{Gr}$$

- Note that $\mathrm{L}G \times^{\mathrm{L}^+G} \mathrm{Gr} = \mathrm{Gr}\tilde{\times}\mathrm{Gr}$ is the convolution Grassmannian.
- Let $X = \mathrm{Gr}_{\leq\lambda}\tilde{\times}\mathrm{Gr}_{\leq\mu}$ be the closure in $\mathrm{Gr}\tilde{\times}\mathrm{Gr}$ of the set of points of the form (gt^λ, ht^μ) with $g, h \in \mathrm{L}^+G$.

A lemma on convolutions (cont)

Proof

- We have $p^{-1}(\mathrm{Gr}_{\leq\lambda} \times \mathrm{Gr}_{\leq\mu}) = q^{-1}(X)$.
- It follows that $p^*(\mathrm{IC}_\lambda \boxtimes \mathrm{IC}_\mu) = q^*(\mathrm{IC}_X)$.
- Thus, by definition, $\mathrm{IC}_\lambda \star \mathrm{IC}_\mu = m_*(\mathrm{IC}_X)$.
- Key observation: m maps X birationally to $\mathrm{Gr}_{\leq\lambda+\mu}$. (This is related to how multiplication works in the Hecke algebra.)
- It follows that $\mathrm{IC}_{\lambda+\mu}$ is a summand of $m_*(\mathrm{IC}_X)$. (This uses a decomposition theorem argument similar to the previous one.)

Remark

If λ and μ are central then $\mathrm{H}^*(\mathrm{IC}_\lambda \star \mathrm{IC}_\mu) = \mathrm{H}^*(\mathrm{IC}_\lambda) \otimes \mathrm{H}^*(\mathrm{IC}_\mu)$ is one-dimensional, so $\mathrm{IC}_\lambda \star \mathrm{IC}_\mu = \mathrm{IC}_\nu$ for some central ν; by the lemma, we must have $\nu = \lambda + \mu$.

First properties of Γ

Proposition

Γ is of finite type, connected, and reductive.

Proof

- By the lemma, finitely many IC_μ's generated Sat_G as a tensor category (e.g., those with μ a fundamental coweight), which implies that Γ is finite type.
- The lemma also shows there is no tensor subcategory of Sat_G containing direct sums of only finitely many IC_μ's. Thus Γ is connected.
- Finally, since Sat_G is semisimple, Γ is reductive.

References

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